# Extremal combinatorics

graph theory : Four color Theorem, Ramsey, random graph

① background

② technique : Combinatorics. number theory. Prob.

③ applications : coding theory. TCS

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Extremal graph theory

finite family $\subset$ $f$. $f(G)$ max min

1. Given $n$ irrational number $x_1, x_2, \cdots, x_n$.

$(x_i, x_j)$ pair. good : $x_i + x_j$ is rational.

determine the maximum number of "good" pairs.

$x_1 \to v_1$, $x_2 \to v_2$, $\cdots$, $x_n \to v_n$

$(x_i, x_j) \to$ good pair $v_i v_j$


$v_1 v_2$
$x_1$

$x_1 + x_2 \in \mathbb{Q}$.

$\left.\begin{array}{l} x_1 + x_3 \in \mathbb{Q} \\ x_2 + x_3 \in \mathbb{Q} \end{array}\right\}$ $x_1 + x_3 - (x_2 + x_3) \in \mathbb{Q}$

$x_1 - x_2 \in \mathbb{Q}$.

$\leq \frac{n^2}{4}$

$x_1 + x_2 + (x_1 - x_2) \in \mathbb{Q}$

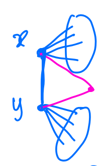
$\therefore 2x_1 \in \mathbb{Q}$

$\triangle$-free $\quad |V(G)| = n.$ maximize $|E(G)|$ ?

$1+\sqrt{2}$ $\quad 1-\sqrt{2}$
$2+\sqrt{2}$
$\vdots$
$10+\sqrt{2}$

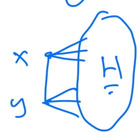
**Theorem 1** (Mantel's theorem) If a graph $G$ on $n$ vertices contains no triangles, then it contains at most $\frac{n^2}{4}$ edges.

**First proof.** We proceed by induction on $n$. For $n=1$ & $n=2$, it holds clearly. $\because$ $\frac{2\times 2}{4} = 1$. We assume that it is true for $(\leq n-1)$ and let $G$ be a graph on $n$ vertices.



Let $x$ and $y$ be two adjacent vertices in $G$.

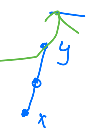
$$d(x)+d(y) \leq n$$



$$e(G) = e(H) + d(x) + d(y) - 1$$
$$\leq \frac{(n-2)^2}{4} + n - 1 = \frac{n^2-4n+4}{4} + n - 1$$
$$= \frac{n^2-4n+4+4n-4}{4} = \frac{n^2}{4}. \quad \square$$

**Second proof.** suppose $\underset{|E(G)|}{e(G)} = m$. If $xy \in E(G)$, then $d(x)+d(y) \leq n$.

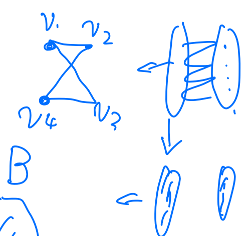
$$\sum_{x\in V(G)} d^2(x) \overset{?}{=} \sum_{xy\in E(G)} \big(d(x)+d(y)\big) \leq \sum_{xy\in E(G)} (n) \leq |E(G)|\cdot n = e(G)\cdot n = mn$$



$\underbrace{d(x)+d(x)+\cdots d(x)}_{d(x)}$ $\quad d^2(x)$

$$\sum_{x\in V(G)} d^2(x) \geq \frac{1}{n}\times\Big(\sum d(x)\Big)^2 \qquad \sum_{x\in V(G)} d(x) = 2e(G) = 2m$$
$$= \frac{1}{n}\times(2m)^2 = \frac{1}{n}\times 4m^2$$
$$\frac{1}{n}\times 4m^2 \leq mn \qquad \therefore\ m \leq \frac{n^2}{4}.$$

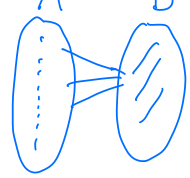
**Third proof.** independent set (独立集).



Let $A$ be the largest independent set in the graph $G$.

$\forall x \in V(G)$: $d(x) \leq |A|$ ?

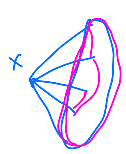
A $\quad$ B



clique

complete bipartite graph.

$N(x)$: independent set

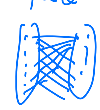


$\triangle$-free

Let $B$ the complement of $A$

$(B = V(G)\setminus A)$

$|A|+|B| = n$



$n/2 \quad n/2$

$$e(G) \overset{?}{\leq} \sum_{x\in B} d(x) \leq |A|\cdot|B| \leq \left(\frac{|A|+|B|}{2}\right)^2 \leq \frac{n^2}{4}.$$

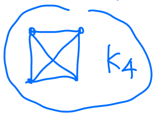
$B$: independent set $\qquad d(x) = |A| \quad \forall x \in B.$

$|A| = |B| = \frac{n}{2}$

$n$ even.

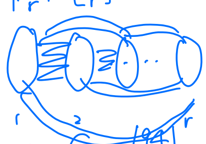
$\triangle$- free $\quad$ $n/2$ $\quad$ $n/2$ $\quad$ $\lceil n/2 \rceil$ $\lfloor n/2 \rfloor$

$K_3$



$K_4$ $\quad$ $K_4$- free $\quad$ $n$-vertex graph. $\quad$ $e(G) \le$ ?

$(K_{r+1})$- free $\quad$ $\lceil \frac{n}{r} \rceil$ $\lfloor \frac{n}{r} \rfloor$



$\left(\frac{n}{3} \times \frac{n}{3}\right) \times 3 = \frac{n^2}{3}$

$(\frac{n}{3})$ $(\frac{n}{3})$ $(\frac{n}{3})$ $\quad$ $r$

$(1-\frac{1}{r}) \times \frac{n^2}{2}$

**Theorem (Turán's theorem)** (Paul Turán)

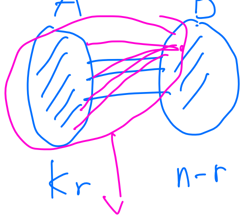
If a graph on $n$ vertices contains no copy of $K_{r+1}$ (the complete graph on $r+1$ vertices), then it contains at most $(1-\frac{1}{r}) \times \frac{n^2}{2}$ edges.

clique $\quad$ Erdős $\quad$ ~~Erdös~~

First proof. By induction on $n$. The theorem is trivially true for $n = 1, 2, \cdots, r$. $\quad$ $\binom{n}{2} \le (1-\frac{1}{r}) \times \frac{n^2}{2}$ $\quad (n \le r)$ $\quad$ $\frac{n(n-1)}{2} \le (1-\frac{1}{r}) \times \frac{n^2}{2}$ ?

We assume that it holds for all values less than $n$ and prove it for $n$.

Let $G$ be a graph on $n$ vertices containing no $K_{r+1}$ and has the maximum possible number of edges. Then $G$ contains a copy of $K_r$ (Otherwise we could add edge to $G$, which contradicts with maximality)



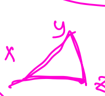
A $\quad$ B $\quad$ $K_r$ $\quad$ $n-r$ $\quad$ $K_{r+1}$

$\forall x \in B$, $\therefore x$ has at most $(r-1)$ neighbors in $A$.

$$
\begin{aligned}
e(G) &\le e_A + e_B + e_{A,B} \\
&\le \binom{r}{2} + (1-\tfrac{1}{r}) \times \frac{(n-r)^2}{2} + (r-1)\cdot(n-r) \\
&\le \frac{r(r-1)}{2} + \frac{(n-r)}{2}\left[\left(1-\tfrac{1}{r}\right) \times (n-r) + 2(r-1)\right] \\
&\le \frac{r(r-1)}{2} + \frac{n-r}{2}\left[\frac{r-1}{r} \times (n-r) + 2(r-1)\right] \\
&\le \frac{r(r-1)}{2} + \frac{r(n-r)\times(r-1)}{2}\left[\frac{n-r}{r} + 2\right] \\
&\le \frac{r(r-1)}{2} + \frac{(r-1)(n-r)}{2} \times \left(\frac{n+r}{r}\right) \\
&= \frac{(r-1)}{2}\left[r + \frac{(n-r)(n+r)}{r}\right] = \frac{r-1}{2} \times \frac{n^2}{r} = \\
&= \frac{r-1}{r} \times \frac{n^2}{2} = (1-\tfrac{1}{r}) \times \frac{n^2}{2}.
\end{aligned}
$$

Second proof.

We assume that $G$ contains no $K_{r+1}$ and has the maximum possible number of edges. We will prove that



$x$ $y$ $z$

if $xy \notin E(G)$ and $yz \notin E(G)$, then $xz \notin E(G)$.

$x$ $y$ $z$

This implies that the graph must be a

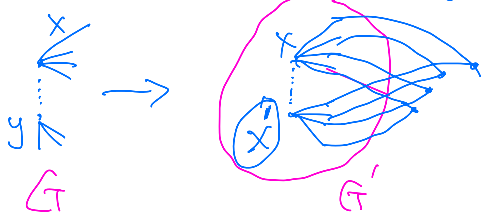
Complete multipartite graph.

Suppose for the contrary that $xy \notin E(G)$, and $yz \notin E(G)$, but $xz \in E(G)$.

Claim

① $d(y) \geq d(x)$ If $d(y) < d(x)$, then we may construct a new graph $G'$ ($K_{r+1}$-free) by deleting $y$ and creating a new copy of $x$, say $x'$.

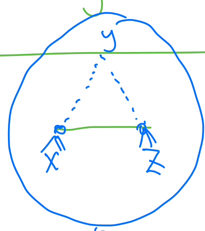
$G'$: $K_{r+1}$ free?



$$|E(G')| = |E(G)| - d(y) + d(x) > |E(G)|$$

② $d(y) \geq d(z)$ If $d(y) < d(z)$, then .. $G''$ by deleting $y$ and adding a new copy of $z'$ ...

$$|E(G'')| = |E(G)| - d(y) + d(z) > |E(G)|. \quad \text{--}$$



We creat a new graph $G_1$ by deleting $x$ and $z$ and creating two extra copies of the vertex $y$.

$G_1$: $K_{r+1}$-free

$$|E(G_1)| = |E(G)| - (d(x) + d(z) - 1) + 2d(y) > |E(G)|.$$

We have a contradiction.

We know that the graph is a complete multipartite graph.

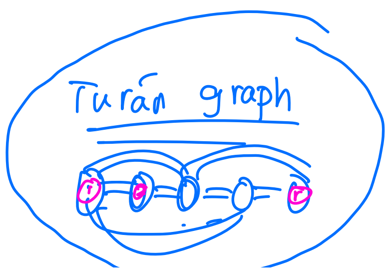
Clearly, it has at most $r$ parts, $\lceil \frac{n}{r} \rceil$ $\lfloor \frac{n}{r} \rfloor$

Homework

① Prove that every $n$-vertex graph with at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

$\Delta$-free $e(G) \leq \frac{n^2}{4}$

$K_{r+1}$-free $e(G) \leq (1 - \frac{1}{r}) \times \frac{n^2}{2}$.



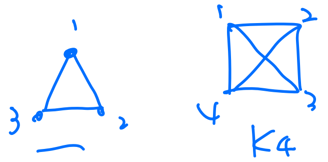
Turán graph: $T_{n,r}$ is defined to be the complete $n$-vertex $r$-partite graph with part sizes differing by at most 1.

(So each part has size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$).   $\chi(T_{n,r}) = r$ ?

$\forall$. $H$:   $H$-free $n$-vertex graph.   $e(G) \leq$ ?

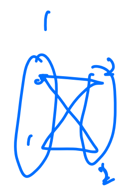
$ex(n, H)$: $\max\{e(G) \mid G$ is $H$-free and $|V(G)| = n\}$

$\Downarrow$ chromatic number of $G$: $\chi(G)$:   $\chi(H)$




$\chi(K_4) = 4$

$\chi(C_4) = 2$



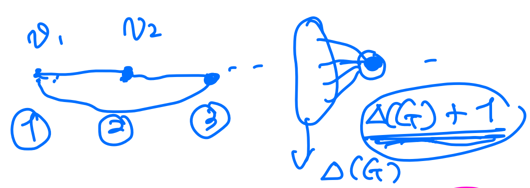

proper $k$-coloring of $G$: $c: V(G) \to \{1, 2, \ldots k\}$ s.t. $\forall\, xy \in E(G)$
$c(x) \neq c(y)$

$\chi(G)$: minimize $k$.   $\chi(K_{r+1}) = r+1$

$\chi(\text{planar graph}) \leq 4$

$G$:   $\Delta(G)$: maximum degree.

$\chi(G) \leq \Delta(G) + 1$ ?   $\chi(G) \leq |V(G)|$




$$ex(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \times \frac{n^2}{2}$$

$H = K_{r+1}$   $1 - \frac{1}{r+1-1} = \left(1 - \frac{1}{r}\right)$   "degenerate"

$\chi(H) = 2$   $\left(1 - \frac{1}{2-1}\right) \times \frac{n^2}{2} =$

$ex(n, C_{2k})$ $\rightsquigarrow$ Regularity lemma.